

On Sylvester waves and restricted partitions

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The higher Sylvester waves are discussed. Techniques used involve finite difference operators. For example, using Herschel's theorem, elegant expressions for Euler's rational functions and the Todd operator are found. Derivative expansions are also rapidly treated by the same method.

A general form for the wave is obtained using multiplicative series, and comments made on its further reduction. As is known, Dedekind sums arise in the case of coprime components and it is pointed out that Brioschi had this result, but not the terminology, very early on. His proof is repeated.

Adding a set of ones to the components of the denumerant corresponds to a succession of (discrete) smoothings and is a Cesaro sum. Using a spectral vocabulary, I take the opportunity to exhibit the finite difference counterparts of some continuous, distributional properties of Riesz typical means.

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1. Introduction

This paper is a continuation of an earlier one, [1], concerned with Sylvester waves, Ehrhart polynomials and degeneracies in spectral problems.² In that work I gave explicit formulae for the first two waves, resurrecting Sylvester's expressions. Here I consider the higher waves but not so far as to produce similarly clear-cut results. However, during the analysis, several amusing pieces of information and technique arose and will be exposed here.

A classic number problem is that of restricted partitions. Given a set of non-negative integers \mathbf{d}_i , ($i = 1, 2, \dots, d$), in how many ways can a given non-negative integer, say l , be expressed as a (non-negative) integer linear combination of the d_i ? I call the d_i , the 'components' and write this combination

$$l = m_i d_i = \mathbf{m} \cdot \mathbf{d}. \quad (1)$$

The number of ways is referred to as the denumerant of l , following Sylvester, who was one of the first to study this problem in any detail. Euler gives the number in terms of a generating function

$$\sum_{l=0}^{\infty} \frac{l;}{\mathbf{d};} \sigma^l = \prod_{i=1}^d \frac{1}{1 - \sigma^{d_i}}, \quad (2)$$

where the coefficient of the σ^l is the denumerant. (I use the notation of [2] to avoid typographical confusion.)

It is frequently possible, in discussions involving the combination (1), to work throughout, more or less, with this generating function. For example, in spectral problems, where the denumerant might come up as a degeneracy, one could set $\sigma = e^{-t}$ and interpret the left-hand side as a cylinder heat-kernel associated, perhaps, with the square root of a Laplace-like operator, after some slight adjustments. A Mellin transform would then yield the corresponding ζ -function which, in this case, would be a Barnes ζ -function,

$$\begin{aligned} \zeta(s, a \mid \mathbf{d}) &\equiv \sum_{l \geq 0} \frac{l;}{\mathbf{d};} \frac{1}{(a+l)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \frac{e^{-at}}{\prod_i (1 - e^{-d_i t})} \\ &= \frac{i\Gamma(1-s)}{2\pi} \int_L dt (-t)^{s-1} \frac{e^{-at}}{\prod_i (1 - e^{-d_i t})}. \end{aligned} \quad (3)$$

² The appearance of the present work has been delayed by factors beyond my control.

(On this definition, the d_i are not restricted to be positive integers.)

I will not pursue this global quantity further now because there is interest in studying the denominator on its own, as attested to by the long history.

Sylvester proves the theorem that the denominator can be expressed as a finite sum of quantities periodic in l , called waves, each of which is associated with a root of unity. That associated with the root 1 is not periodic (has period infinity) and is a normal polynomial, W_1 , in l . Hence one can set,

$$\frac{l;}{\mathbf{d};} = W_1 + U$$

where U is the properly periodic part and is a sum of waves, generally written

$$U = W_2 + \sum_{q>2} W_q. \quad (4)$$

The whole denominator is a quasipolynomial in l *i.e.* a polynomial with periodic coefficients, *e.g.* [3]. A simple proof of this was given by Wright, [4].

As a function of the augmented variable, $\bar{l} \equiv l + \sum_i d_i/2$, the denominator $l; / \mathbf{d};$ satisfies a parity reciprocity under $\bar{l} \rightarrow -\bar{l}$, [5]. The choice between l and \bar{l} comes up later.

The second wave W_2 , is that associated with the root -1 , and has also been separated in (4) because Sylvester has given an explicit form for W_1 and a more or less explicit form for W_2 . A simpler expression for W_2 , like that for W_1 , was presented in our earlier work, [1], and I wish here to treat the other waves by a similar technique.

The evaluation of the denominator is a linear Diophantine question related to the Frobenius problem and to the Ehrhart polynomial and has thus been the subject of extensive analysis. I refer to two books, [6], [7], for some history and motivation. However, Sylvester's specific formula is not referred to very often. The papers by Rubinstein and Fel, [8], and Rubinstein, [9], describe his method, but the detailed techniques are different. A discussion of Sylvester's basic method can be found in the book by Netto, [10] published first in 1902, although he does not give Sylvester's final form for W_1 , which seems to be largely ignored, apart from the 1909 paper of Glaisher, [11], who extends the working to the other waves. The present paper could be looked upon as an independent, partial recasting of Glaisher's computations in as small a compass as I could manage, plus some comments about the literature.³

There is some very interesting recent work on computing these partition numbers [12], [13].

³ Unfortunately, I have not been able to see some relevant early Italian work by Trudi.

2. Sylvester's waves

Sylvester's theorem leads to a prescription for the waves W_q which is the following. Write out all the factors of the components, d_i . Let there be μ such and call a typical one, q . Separate the components into two groups – those divisible by q , call them α_i , $i = 1, 2, \dots, \alpha$ (α is the 'frequency') and those that are not, say β_j , $j = 1, 2, \dots, \beta$ with $\alpha + \beta = \mu$. Sylvester, [14], then says that,

$$W_q = \text{co}_{-1} \sum_{\rho} \frac{\rho^{\bar{l}} e^{\bar{l}t}}{\prod_i (e^{\alpha_i t/2} - e^{-\alpha_i t/2}) \prod_j (\rho^{\beta_j/2} e^{\beta_j t/2} - \rho^{-\beta_j/2} e^{-\beta_j t/2})} \quad (5)$$

which stands for the coefficient of $1/t$ in the indicated generating function. The sum is over all *prime* q th roots of unity, ρ , *i.e.* $\rho^q = 1$, $\rho = e^{2\pi i p/q}$, $(p, q) = 1$. The variable \bar{l} is the augmented one, $\bar{l} \equiv l + \sum \alpha_i/2 + \sum \beta_j/2 = l + \sum d_i/2$.

The necessary inverse power(s) of t are provided only by the α_i group. Therefore, expanding the exponential, the coefficient of $\bar{l}^n/n!$ is

$$\begin{aligned} & \text{co}_{-1} \sum_{\rho} \frac{\rho^l t^n}{\prod_i (e^{\alpha_i t/2} - e^{-\alpha_i t/2}) \prod_j (e^{\beta_j t/2} - \rho^{-\beta_j} e^{-\beta_j t/2})} \\ &= \frac{1}{\prod \alpha_i} \sum_{\rho} \frac{\rho^l}{\prod_j (1 - \rho^{-\beta_j})} \text{co}_{\alpha-n-1} \prod_i \frac{\alpha_i t/2}{\sinh \alpha_i t/2} \prod_j \frac{\sin \beta_j \pi p/q}{\sin \beta_j (\pi p/q - it/2)}. \end{aligned} \quad (6)$$

So far as the exponent of ρ is concerned, it is best to retain the integer l , as Netto [10] suggests, also $\rho^{q-1} + \rho^{q-2} + \dots + 1 = 0$.

As in my earlier work, I invoke some of Euler's old products. Specifically, after taking logs

$$\log \frac{z}{\sinh z} = - \sum_{n=1}^{\infty} \log \left(1 + \frac{z^2}{n^2 \pi^2} \right) = - \frac{\zeta(2)}{\pi^2} z^2 - \frac{\zeta(4)}{2\pi^4} z^4 - \frac{\zeta(6)}{3\pi^6} z^6 - \dots \quad (7)$$

and

$$\begin{aligned}
\log \frac{\sin \pi a}{\sin \pi(z+a)} &= - \sum_{n=1}^{\infty} \log \left(1 - \frac{z}{n-a} \right) + - \sum_{n=0}^{\infty} \log \left(1 + \frac{z}{n+a} \right) \\
&= - \lim (\zeta(1, a) - \zeta(1, 1-a))z + \frac{1}{2}(\zeta(2, a) + \zeta(2, 1-a))z^2 - \\
&\quad \frac{1}{3}(\zeta(3, a) - \zeta(3, 1-a))z^3 + \frac{1}{4}(\zeta(4, a) + \zeta(4, 1-a))z^4 - \\
&= -z\pi \cot \pi a - \frac{1}{2}z^2 \frac{d}{da} \pi \cot \pi a - \frac{1}{3} \frac{1}{2!} z^3 \frac{d^2}{da^2} \pi \cot \pi a - \\
&= - \left(\int_0^z dz e^{z \frac{d}{da}} \right) \pi \cot \pi a = \frac{1 - e^{z \frac{d}{da}}}{\partial_a} \pi \cot \pi a \\
&= - \int_0^z dz \pi \cot \pi(z+a)
\end{aligned} \tag{8}$$

which is really only a check because it follows directly by school calculus.

Reversing the argument, the step from the second to the third line, which is the standard reflection formulae of the Hurwitz ζ -function, (or, equivalently of the polygamma function),

$$\zeta(n, a) + (-1)^n \zeta(n, 1-a) = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} \pi \cot \pi a,$$

can be derived. For this nexus of notions consult Hoffman, [15,16], who has a different organisation, and a little history.

I also note that one could expand in terms of the Euler rational functions, *e.g.* Carlitz [17], [8], defined by,

$$\frac{1-\lambda}{\lambda-e^x} = \sum_n H_n(\lambda) \frac{x^n}{n!}. \tag{9}$$

As a practical arithmetical means of calculation in any specific numerical case when the number of components and the frequencies are not too large, this is probably as good a route as any. It was used by Cayley in his treatment of partitions, [18]. He gives two (related) relevant expansions, based on Herschel's theorem, *e.g.* [19], [20], [21]. Firstly, for small denumerants,

$$\begin{aligned}
\frac{1}{1-c e^{-t}} &= \sum_{f=0}^{\infty} \frac{(-1)^f}{f!} t^f \frac{1}{1-c(1+\Delta)} 0^f \\
&= \frac{1}{1-c(1+\Delta)} e^{-0.t}
\end{aligned} \tag{10}$$

which gives a direct expression for the Euler functions (9) in finite difference terms. The quantity acting on 0^f is a finite polynomial in the difference operator, Δ , because $\Delta^n 0^m = 0$ if $n > m$ and, generally,

$$\Delta^n 0^m = (-1)^n \sum_{k=0}^n (-1)^k k^m \binom{n}{k} = n! \left\{ \begin{matrix} m \\ n \end{matrix} \right\}, \quad (11)$$

in terms of Stirling numbers of the second kind, a very old result. Thus I find the elegant expression,

$$H_n(\lambda) = - \left(1 - \frac{1}{\lambda-1} \Delta \right)^{-1} 0^n, \quad (12)$$

which I do not take any further here but note that it is equivalent to a result of Vandiver, [22]. See later.

Moreover, a form for the Todd operator, $\text{Todd}(c, \partial_h)$, of Brion and Vergne, [23], is

$$\frac{\partial}{1 - c e^{-\partial_h}} = \frac{\log(1 + \Delta)}{c(1 + \Delta) - 1} e^{-0.\partial_h}, \quad (13)$$

so that

$$\text{Todd}(c, \partial_h) f(h) = \frac{\log(1 + \Delta)}{c(1 + \Delta) - 1} f(h - 0). \quad (14)$$

For complicated denumerants, the logarithm is best expanded using (10) and

$$\begin{aligned} \log \frac{1-c}{1-c e^{-t}} &= \int_0^t dt \left(1 - \frac{1}{1-c e^{-t}} \right) \\ &= \frac{1}{1-c(1+\Delta)} \frac{1}{0} (1 - e^{-0.t}). \end{aligned}$$

These particular expressions lead to the denumerant expressed in terms of the unaugmented variable, l , and this is what Cayley produces after some calculation. Interesting work on relieving some of this effort is given by Munagi, [13], and Sills and Zeilberger, [12].

To encounter the (preferred) augmented variable, \bar{l} , a more symmetrical expansion is required, corresponding to the cotangent in (8). The relevant expansion follows as

$$\frac{c e^{-t}}{1 - c e^{-t}} = \frac{c(1 + \Delta)}{1 - c(1 + \Delta)} e^{-0.t}. \quad (15)$$

Set, now, $c = e^{-s}$ so that the left-hand side becomes

$$\frac{e^{-s-t}}{1 - e^{-s-t}}$$

and the derivatives with respect to t , at $t = 0$, which are explicit in (15), are just the derivatives of $e^{-s}/(1 - e^{-s}) = (\coth s/2 - 1)/2$ with respect to s . This, incidentally, provides an elegant difference proof of these higher derivatives,

$$\begin{aligned}
\frac{1}{2} \frac{d^n}{ds^n} \left(\coth s/2 - 1 \right) &= \frac{d^n}{ds^n} \frac{1}{e^s - 1} = (-1)^n \frac{1 + \Delta}{e^s - (1 + \Delta)} 0^n \\
&= -(-1)^n \log \left(1 - \frac{\Delta}{e^s - 1} \right) 0^{n+1} \\
&= (-1)^n \sum_{k=1}^{n+1} \frac{1}{k} \frac{1}{(e^s - 1)^k} \Delta^k 0^{n+1} \\
&= (-1)^n \sum_{k=1}^{n+1} (k-1)! \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \frac{1}{(e^s - 1)^k},
\end{aligned} \tag{16}$$

using (11) to give an explicit expression. The passage from the first to the second line is effected by the identity due to Herschel (Boole, [24], Ch.II, §13),

$$\phi(\Delta) 0^{n+1} = (1 + \Delta) \phi'(\Delta) 0^n,$$

whose essential content is the recurrence relation for the Stirling numbers.

The expansion, (16) and its equivalents, surface from time to time. Agoh and Dilcher, [25], prove it by induction and use it, in several papers, to derive various Bernoulli number identities and Sterling convolutions.

It is equivalent to a formula in Adamchik, [26]. See also Cvijović, [27], Knopf, [28], Hoffman, [15], Boyadzhiev, [29]. The technical relation used in these works is,

$$\left(x \frac{d}{dx} \right)^n f(x) \equiv \frac{\overline{d^n}}{dx^n} f(x) = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k \frac{d^k}{dx^k} f(x), \tag{17}$$

which Knopf, attributes, in essence, to Scherk in 1824 who, so it seems to me, was addressing the same question solved by Herschel in 1816, arriving at the same result ⁴. Equation (17) is contained in equation (4) in Herschel, [19], and explicitly exhibited as Exercise 6, in Boole, [20], p.26, *viz.*

$$\frac{\overline{d^n}}{dx^n} = \sum_{k=1}^n \frac{1}{k!} \Delta^k 0^n x^k \frac{d^k}{dx^k}. \tag{18}$$

⁴ Gould gives further interesting detail and earlier history in [30].

For amusement, I interject a proof of this. From Herschel's theorem one has,

$$f(e^{t+s}) = f(e^s(1 + \Delta))e^{0.t}, \quad (19)$$

and note the oft used device,

$$\left. \frac{d^n}{dt^n} f(e^{t+s}) \right|_{t=0} = \frac{d^n}{ds^n} f(e^s) = \frac{\overline{d^n}}{dx^n} f(x), \quad \text{where } x = e^s,$$

the left-hand side of which can be picked out of (19) as the coefficient of $t^n/n!$. The expansion of $f(x(1 + \Delta))$ in powers of Δ ,

$$f(x(1 + \Delta)) = \sum_{k=0}^{\infty} \frac{1}{k!} \Delta^k 0^n x^k \frac{d^k}{dx^k} f(x),$$

after retaining powers of Δ no greater than n yields (18). (The first term also goes away because $0^n = 0$.)

These considerations also show the equivalence of (12) with a formula in Vandiver, [22], who also employs the operator \overline{d}/dx .

After this digression, I return to the computation of the denominator, *i.e.* (6), where one has to deal with the products. The first product has been encountered in [1] and so I turn attention to the second one, to which (8) applies after the identifications, $a = a_j = \beta_j p/q$ and $z = z_j = -i\beta_j t/2$. I remark that πa_j is half the argument of ρ^{β_j} , $\rho^{\beta_j} = \exp(2\pi i a_j)$

Denoting by $\Omega(z, a)$ the argument of the logarithm in (8), the corresponding multiplicative sequence follows on first constructing the product,

$$\Omega_1 \Omega_2 \dots = \exp \left(-\frac{t}{2} \Xi_1 + \frac{1}{2} \frac{t^2}{2^2} \Xi_2 - \frac{1}{3} \frac{t^3}{2^3} \frac{1}{2!} \Xi_3 + \frac{1}{4} \frac{t^4}{2^4} \frac{1}{3!} \Xi_4 + \dots \right) \quad (20)$$

where the Ξ_n are defined by (8).

$$\begin{aligned} \Xi_n &= \sum_j i^n \beta_j^n \frac{d^{n-1}}{da^{n-1}} \cot \pi a \Big|_{a=a_j} \\ &= 2 \sum_j (-1)^n \beta_j^n \frac{1}{1 - \rho^{-\beta_j} (1 + \Delta)} 0^{n-1}, \quad n > 1 \\ \Xi_1 &= \sum_j \beta_j \frac{1 + \rho^{\beta_j}}{1 - \rho^{\beta_j}} \end{aligned} \quad (21)$$

after employing (16) or its equivalent.

In terms of Stirling numbers, *cf* (16), (21) reads,

$$\Xi_n = 2(-2\pi)^{n-1} \sum_j \beta_j^n \frac{1}{1 - \rho^{\beta_j}} \sum_{k=1}^{n-1} k! \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \left(\frac{\rho^{\beta_j}}{1 - \rho^{\beta_j}} \right)^k, \quad n > 1. \quad (22)$$

Things can be expressed equivalently in terms of known Euler functions. Vandiver has shown that, generally,

$$\frac{\overline{d}^n}{dx^n} \left(\frac{x}{1-x} \right) = \frac{x}{1-x} H_n(x)$$

in terms of the functions defined by (9) a few examples being $H_0 = 1$ and,

$$H_1(x) = \frac{1}{x-1}, \quad H_2(x) = \frac{1-x}{(x-1)^2}, \quad H_3(x) = \frac{1+4x+x^2}{(x-1)^3}. \quad (23)$$

Setting $x = \rho^\beta$ one has that,

$$\left(\frac{\overline{d}}{d\rho^\beta} \right)^n \left(i \cot \pi a - 1 \right) = \frac{2\rho^\beta}{1 - \rho^\beta} H_n(\rho^\beta)$$

Then,

$$\Xi_n = 2(2\pi)^{n-1} \sum_j \beta_j^n \frac{\rho^{\beta_j}}{(1 - \rho^{\beta_j})^n} R_{n-1}(\rho^{\beta_j}), \quad n > 1, \quad (24)$$

where $R_n(x)$ are Euler polynomials, the numerators of (23). This is just (21). A small simplification occurs if the inversion property,

$$R_n(x) = (-1)^{n-1} x^{n-1} R_n(1/x)$$

is used. Then,

$$\Xi_n = 2(2\pi)^{n-1} \sum_j \beta_j^n \frac{1}{(1 - \rho^{-\beta_j})^n} R_{n-1}(\rho^{-\beta_j}), \quad n > 1, \quad (25)$$

which can be obtained directly.

Any systematic analysis of Sylvester's theorem is bound to lead to similar quantities and Glaisher, in his extensive treatment, encounters similar polynomials. See [11], especially §§79,80,93-100. He also employs the difference operator Δ .

Equation (25) yields an explicit, but unsimplified, formula for the product (20), occurring in (6), which now has to be combined with the product over the α_i coming via (7) as

$$\begin{aligned} Q_1 Q_2 \dots &= \exp \left(-s_2 \frac{\zeta(2)}{\pi^2} \frac{t^2}{2^2} + s_4 \frac{\zeta(4)}{2\pi^4} \frac{t^4}{2^4} - s_6 \frac{\zeta(6)}{3\pi^6} \frac{t^6}{2^6} + \dots \right) \\ &\equiv \exp \left(-\frac{\bar{\tau}_2}{2} t^2 + \frac{\bar{\tau}_4}{4} t^4 - \frac{\bar{\tau}_6}{6} t^6 + \dots \right) \end{aligned}$$

where s_n is the sum of the n th powers of the α_i set.⁵

Combining the two series gives,

$$\Omega_1 \Omega_2 \dots Q_1 Q_2 \dots = \exp \left(-\kappa_1 t + \frac{1}{2} \kappa_2 t^2 - \frac{1}{3} \kappa_3 t^3 + \dots \right)$$

where

$$\begin{aligned} \kappa_1 &= \frac{1}{2} \Xi_1 \\ \kappa_2 &= \frac{1}{2^2} \Xi_2 - \bar{\tau}_2 \\ \kappa_3 &= \frac{1}{2^3 2!} \Xi_3 \\ \kappa_4 &= \frac{1}{2^4 3!} \Xi_4 + \bar{\tau}_4 \\ &\text{etc.} \end{aligned} \tag{26}$$

The final step expands the exponential as a power series

$$\Omega_1 \Omega_2 \dots Q_1 Q_2 \dots = 1 - \Theta_1(\kappa_1) t + \Theta_2(\kappa_1, \kappa_2) t^2 - \Theta_3(\kappa_1, \kappa_2, \kappa_3) t^3 + \dots$$

where the $\Theta_r([\kappa]_r)$ are all the homogeneous products of the quantities of which the κ_i would be sums of powers and are classic functions of the κ_i , (see [5], [1]). Some examples are

$$\Theta_0 = 1, \quad \Theta_1 = \kappa_1, \quad \Theta_2 = \frac{1}{2}(\kappa_1^2 + \kappa_2), \quad \Theta_3 = \frac{1}{6}(\kappa_1^3 + 3\kappa_1\kappa_2 + 2\kappa_3). \tag{27}$$

Leaving the expressions as they are, returning to (6) for the polynomial coefficient, the polynomial for the wave W_q takes the finite form

$$W_q = \frac{1}{\prod \alpha_i} \sum_{\rho} \frac{\rho^l}{\prod_j (1 - \rho^{-\beta_j})} \left(\frac{\bar{l}^{\alpha-1}}{(\alpha-1)!} \Theta_0 - \frac{\bar{l}^{\alpha-2}}{(\alpha-2)!} \Theta_1 + \frac{\bar{l}^{\alpha-3}}{(\alpha-3)!} \Theta_2 - \dots \right). \tag{28}$$

⁵ The relation between the constants, $\bar{\tau}$, here and those, τ , in [1] and [5], is $\bar{\tau}_{2n} = 2\tau_n$. This reflects the fact that the series now contains all powers of t .

A trivial check is provided by W_2 , for which $\rho = -1$ and so, since the β_j are all odd, the preliminary factor in (28) is just $1/(2^\beta \prod \alpha)$. All the Ξ_n are zero and the result, using (26), reduces to that in [1,11].

In this case the roots of unity dependence is trivial, which is not true for the general wave and this is the remaining computational issue. A strategic decision has to be taken concerning the ‘final’ form for the periodic polynomial. Sylvester in [2] writes it in terms of elementary denumerants, $(\bar{l} \pm r); /q;$, which first entails a reduction into a polynomial in the prime roots, ρ . This reduction also occurs in the calculations of Cayley, [31] p.50, who expresses the final answer, equivalently, in terms of prime circulators, as does Glaisher, whose calculation of W_5 , [11] §§81-87, which he takes to the third term in (28) involving the square of Ξ_1 , brings out the attendant complications. Andrews, [32], rewrites things in terms of the greatest integer function, which makes the integrality more obvious, it being obscured in the other formulations. In, *e.g.* Beck and Robins, [6], the roots of unity expressions are not taken further but are mostly left, and analysed, as (generalised) Dedekind sums.⁶

3. The simplest case of coprime components. Dedekind sums

There is no need for complicated bookkeeping when the components, d_i , are mutually prime. There is a subset of waves belonging to each factor on the denominator, separately.

The expression for W_1 was given by Sylvester, [5], and was rederived in [11,1] so I consider the higher waves, W_q , where q divides the typical component d_i , $i = 1, \dots, d$, also treated in [11]. Rather than write out this case specially, it is easier to refer to the general form (5), set $\alpha = 1$, $\alpha_1 = d_i$. There is only one term on the denominator that is proportional to t so that t can be set equal to 0 in the remaining terms. This easily gives,

$$W_q = \frac{1}{q} \sum_{\rho} \frac{\rho^l}{\prod_j (1 - \rho^{-\beta_j})} = \frac{1}{q} \sum_{\rho} \frac{\rho^{-l}}{\prod_j (1 - \rho^{\beta_j})}, \quad (29)$$

by setting $\rho \rightarrow \rho^{-1}$. The β_i are the remaining components and the ρ are the non-trivial q -prime roots of unity, of which there are $q - 1$, if q happens to be prime

⁶ Beck and Robins note that Dedekind sums occur, implicitly, in the work of Sylvester and explicitly in Israilov, [33] in 1979.

when $\rho_m = e^{2\pi m/q}$ and also, in this case,

$$W_q = \frac{1}{q} \sum_{m=1}^{q-1} \frac{\rho_m^l}{\prod_j (1 - \rho_m^{-\beta_j})} = \frac{1}{q} \sum_{m=1}^{q-1} \frac{\rho_m^{-l}}{\prod_j (1 - \rho_m^{\beta_j})}. \quad (30)$$

These W_q are just the Fourier–Dedekind sums studied in [6], and earlier references therein. The actual definition there used is,

$$s_l(\beta; q) \equiv \frac{1}{q} \sum_{m=1}^{q-1} \frac{\rho_m^l}{\prod_j (1 - \rho_m^{\beta_j})},$$

so that it is only notation to write the wave, (30), as,

$$W_q = s_{-l}(\beta; q).$$

The special case expression, (29), occurs in the early short note by Brioschi, [34], equn.(8), and so one should add his name to those who had already encountered Dedekind sums. He also gives a contour proof of the general Sylvester theorem and I repeat it in the Appendix for the coprime case.

Leaving the waves as Dedekind sums, (29) cannot be considered as a complete determination and some energy has to be expended in computing the sums. This is the purpose of the manipulations of Cayley, Glaisher and Sylvester, amongst others. Brioschi, [34], works out the particular example $\mathbf{d} = (2, 3, 5)$. Sylvester, [2], does $(1, 2, 3)$, $(1, 4, 7)$ and $(1, 3, 5)$, for those who like variety. Remarks on the computability of the Dedekind sums are also made in [6] and *cf* Exercise 8.2.

I consider the factor $\prod_j (1 - \rho^{-\beta_j})^{-1}$ where, to remind, ρ is a primitive q th root of unity and the β_j are the components of the denominator not divisible by q , β in number. It can be advantageous, as suggested by Cayley, [31], and done by Glaisher, [11], to reduce the $\beta_j \bmod q$. Then

$$\prod_j (1 - \rho^{-\beta_j})^{-1} = \prod_{m=1}^{q-1} (1 - \rho^{-m})^{-h_m}$$

where the non-negative integers h_m depend on the set of the β_j and, in general, have no pattern.

4. Roots of unity and prime circulators

No point would be served by continuing the calculation along Glaisher's lines, which rapidly becomes unwieldy. Something more universal and automatic is required. Unfortunately I cannot provide it here.

For example, for W_q , Glaisher reduces, mod q , the powers, β_j of ρ before performing any combinations. In a general method this would be premature and involve unnecessary extra labour.

Looking at the general structure (28), with (27), (26) and (21) or (25), it is sufficient to write each term in the summation over j in (25) as a polynomial in ρ of order $q - 1$ (using $\rho^q = 1$) and then reduce any products of these (coming from the powers in (26)) to similar polynomials. Ultimately, the Θ_k are then also such polynomials and one then converts $\Theta_k / \prod_j (1 - \rho^{-\beta_j})$ to a like polynomial, which can then be converted into Cayley's prime circulators, if desired.

The basic algebraic problem is to reduce the ratio of polynomials of a primitive q th root of unity, ρ_μ , to a similar polynomial,

$$\frac{\sum_{m=0}^{q-1} A_m \rho_\mu^m}{\sum_{m=0}^{q-1} B_m \rho_\mu^m} = \sum_{m=0}^{q-1} C_m \rho_\mu^m,$$

cf Battaglini (1857). Unfortunately I have not been able to see this work. It is mentioned with a few details in Dickson, [35] p.121.

At this rather unsatisfactory point I leave this detailed algebraic aspect.

5. Denumerants and Cesàro sums

For a given set of components, the numerical computation of a denumerant can proceed in several ways. The classic special case when $d_1 = 1, d_2 = 2$ *etc.*, is treated at length in Gupta [36], where some history is also given. The expressions derived by Cayley (which are essentially the same as Sylvester's as enlarged by Glaisher, [11]) are considered. Large l and largish d are discussed and a typical example is detailed. Cancellations and factorisations occur.

Euler computed many values using recursion, but for smallish l and d it is, perhaps, just as easy to employ a convolution–iteration technique which yields an expression in terms of simple denumerants only.⁷

⁷ The convolution corresponds to the product form of the generating function, (2), and the computation to the time-honoured one of expanding each factor and collecting terms.

The last component d_d can be separated using the convolution

$$\frac{l;}{d_1, d_2, \dots, d_d;} = \sum_{l'=0}^l \frac{(l-l');}{d_d;} \frac{l';}{d_1, d_2, \dots, d_{d-1};}, \quad (31)$$

which is a special case of the more general,

$$\frac{l;}{d_1, d_2, \dots, d_d;} = \sum_{l'=0}^l \frac{(l-l');}{d_j, \dots, d_d;} \frac{l';}{d_1, d_2, \dots, d_{j-1};}. \quad (32)$$

Equation (31) can be iterated to the intermediate form,

$$\frac{l;}{d_1, d_2, \dots, d_d;} = \sum_{l_d=0}^l \frac{(l-l_d);}{d_d;} \sum_{l_{d-1}=0}^{l_d} \frac{(l_d-l_{d-1});}{d_{d-1};} \dots \sum_{l_{d_{i+1}}=0}^{l_{d_i+2}} \frac{(l_{d_{i+2}}-l_{d_{i+1}});}{d_{i+1};} \frac{l_{d_{i+1}};}{d_1, \dots, d_i;} \quad (33)$$

and, completely, down to

$$\frac{l;}{d_1, d_2, \dots, d_d;} = \sum_{l_d=0}^l \frac{(l-l_d);}{d_d;} \sum_{l_{d-1}=0}^{l_d} \frac{(l_d-l_{d-1});}{d_{d-1};} \dots \sum_{l_2=0}^{l_3} \frac{(l_3-l_2);}{d_2;} \frac{l_2;}{d_1;}. \quad (34)$$

The simple denominator, $l;/q;$, is 1 if q divides l and zero otherwise. I refer to this as Herschel's function as it is just the average of the l th powers of all of the q th roots of unity, which Herschel used when introducing his circulating functions, [37]. In terms of the fractional part and the Kronecker delta,

$$\frac{l;}{q;} = \delta_{\{l/q\}, 0},$$

and (34) is very easily programmed but, being recursive, is not very efficient. However it does present the denominator as an obvious integer.⁸

From the spectral aspect, $l;/q;$, $l = 0, 1, \dots$ is the Laplacian (Neumann) degeneracy on the $2q$ -divided circle (or, equivalently, on a π/q interval) (eigenfunctions, $\cos(lq\theta)$). This can be envisaged as a one-dimensional lune.

The convolution, (31), can be thought of as the addition of another component. If the component 1 is added to the simple denominator, $l;/q;$, one obtains $l;/1, q;$ which is the degeneracy on the two-dimensional lune, $\lfloor l/q \rfloor + 1$.

⁸ By this ancient brute force method, I computed $100;/1, 2, 3, 4, 5;$ as 46262 in 2 minutes using DERIVE and an Athlon IIX4 610e processor.

This can be repeated to give the degeneracy on the d -lune and I pursue this particular process, but in the context of the intermediate form (33) which yields,

$$\frac{l;}{d_1, \dots, d_{j-1}, \mathbf{1}_{i-j+1};} = \sum_{l_i=0}^l \sum_{l_{i-1}=0}^{l_i} \cdots \sum_{l_j=0}^{l_{j+1}} \frac{l_j;}{d_1, \dots, d_{j-1};}, \quad (35)$$

which I think of as a succession of smoothings of the summand or as a nested series of accumulated degeneracies.

All summations can be performed except the last, and one finds

$$\frac{l;}{d_1, \dots, d_{j-1}, \mathbf{1}_{i-j+1};} = \sum_{l'=0}^l \binom{i-j+l-l'}{l-l'} \frac{l';}{d_1, \dots, d_{j-1};}, \quad (36)$$

which is, to check, the same as (36), *i.e.* ,

$$\frac{l;}{d_1, \dots, d_{j-1}, \mathbf{1}_{i-j+1};} = \sum_{l'=0}^l \frac{(l-l');}{\mathbf{1}_{i-j+1};} \frac{l';}{d_1, d_2, \dots, d_{j-1};}, \quad (37)$$

in view of the classic value of the unit denominator,

$$\frac{l;}{\mathbf{1}_d;} = \binom{d-1+l}{l},$$

as follows, for example, from the expansion of the generating function,

$$\sum_{l=0}^{\infty} \frac{l;}{\mathbf{1}_d;} \sigma^l = \frac{1}{1-\sigma^d}. \quad (38)$$

This denominator gives the degeneracy on the d -hemisphere.

The generating function version of (37) is the rather trivial splitting,

$$\frac{1}{(1-\sigma)^{i-j+1}(1-\sigma^{d_1}) \cdots (1-\sigma^{d_{j-1}})} = \frac{1}{(1-\sigma)^{i-j+1}} \frac{1}{(1-\sigma^{d_1}) \cdots (1-\sigma^{d_{j-1}})}. \quad (39)$$

It will be recognised that (36) is a Cesàro sum originally introduced to deal with divergent series. The standard situation (see *e.g.* Hobson, [38], Bromwich, [39], Knopp, [40]) is that for an infinite sequence, g_ν , one defines the finite sums

$$S_n^{(r)} = \sum_{\nu=0}^n \binom{r+n-\nu}{r} g_\nu \quad (40)$$

and investigates the r th ‘arithmetic mean’

$$\binom{r+n}{n}^{-1} S_n^{(r)}$$

as $n \rightarrow \infty$. In this expression, the index, r , although it originates as the number of smoothing summations, can assume any value, real or complex. Negative integers, however, are usually excluded (but see later).

In terms of generating functions, (40) translates into

$$\sum_{\nu=0}^{\infty} S_{\nu}^{(r)} \sigma^{\nu} = \frac{1}{(1-\sigma)^{r+1}} \sum_{\nu=0}^{\infty} g_{\nu} \sigma^{\nu} \quad (41)$$

of which (39) is an example.

The Cesàro mean is a discrete analogue of, and a motivation for, the more powerful ‘typical mean’ of Riesz, which introduces a handy continuous variable into the analysis, *e.g.* Hardy and Riesz, [41], [38]. I give only the briefest details in a more general setting.

Typically, in an eigenproblem, ν would be an eigen*level* label, and g_{ν} its degeneracy. A function $N(\lambda)$ of the continuous variable, λ , is defined as a counting function, encoding the spectrum λ_{ν} ,

$$N(\lambda) = \sum_{\nu, \lambda \leq \lambda_{\nu}} g_{\nu}.$$

In the special situation above, ν would be l , g_{ν} the denominator, l/\mathbf{d} ; and the eigenvalue, λ_{ν} , a function of l , typically $a + l$, where a is a constant.

Defining the (first) accumulated degeneracy by,

$$G_{\nu} = \sum_{\nu'=0}^{\nu} g_{\nu'}$$

or,

$$\sum_{\nu=0}^{\infty} G_{\nu} \sigma^{\nu} = \frac{1}{1-\sigma} \sum_{\nu=0}^{\infty} g_{\nu} \sigma^{\nu},$$

one has the connection,

$$\begin{aligned} N(\lambda) &= G_{\nu-1}, \quad \lambda_{\nu-1} < \lambda < \lambda_{\nu} \\ &= G_{\nu-1} + \frac{1}{2} g_{\nu} = G_{\nu+1} - \frac{1}{2} g_{\nu}, \quad \lambda = \lambda_{\nu}, \end{aligned}$$

and, of course, G_ν is the first Cesàro sum $S_\nu^{(0)}$, (40).

Unlike the accumulated degeneracies, $N(\lambda)$ depends on the actual form of the eigenvalues, λ_ν .

The Cesàro generating function, (41) in terms of G , is

$$\sum_{\nu=0}^{\infty} S_\nu^{(r)} \sigma^\nu = \frac{1}{(1-\sigma)^r} \sum_{\nu=0}^{\infty} G_\nu \sigma^\nu. \quad (42)$$

For the *Riesz* mean, this iterated summation of G_ν is replaced by an $(r+1)$ -fold iterated integration of $N(\lambda)$ which equals the Cauchy convolution, (see Knopp, [40]),

$$N_{r+1}(\lambda) = \int_0^\infty d\lambda' \frac{(\lambda - \lambda')^r}{\Gamma(r+1)} N(\lambda'),$$

employed specifically as a smoothing of N by Fedosov, [42], see Baltes and Hilf, [43], Balian and Bloch [44].

Introducing the distribution Φ , (see Gel'fand and Shilov, [45] §5.5),

$$\Phi_\alpha(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)},$$

the convolution is neatly written,

$$N_\alpha = N * \Phi_\alpha,$$

at the same time extending $r+1$ to an arbitrary variable, α , by continuation.

Comparing this convolution with the Cesàro discrete version, (40) or (36), leads to the analogy

$$\binom{l+\alpha-1}{l} = \frac{\Gamma(l+\alpha)}{\Gamma(l+1)\Gamma(\alpha)} \sim \Phi_\alpha(l). \quad (43)$$

For typographical reasons, the left-hand side is often denoted by $A_l^{\alpha-1}$.

It is interesting to enlarge upon this analogy and to exhibit some continuous relations together with their finite difference counterparts.

The distribution Φ has a number of basic properties, [45].

- (i) The concentration of $\Phi_\alpha(x)$ on the positive x axis.
- (ii) The convolution,

$$\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}.$$

- (iii) The Laplace transform,

$$\int_0^\infty dx \Phi_\alpha(x) e^{-tx} = \frac{1}{t^\alpha}, \quad t > 0.$$

(iv) The singularities,

$$\Phi_{-k}(x) = \delta^{(k)}(x).$$

Property (i) corresponds to the statement that $A_l^{\alpha-1} = 0$ if l is a negative integer (because of the poles in the $\Gamma(l+1)$ -function) unless α is a positive integer, when it vanishes at a finite number of negative integers. α a negative integer comes under property (iv).

The convolution property (ii) is a simple consequence of the definition and analytic continuation. The corresponding discrete equation *viz*, the well known binomial identity,

$$\sum_{\nu=0}^l A_{\nu}^{\alpha-1} A_{l-\nu}^{\beta-1} = A_l^{\alpha+\beta-1},$$

follows likewise, most easily from factorisation of the generating function. This is just Cauchy's product.

Property (iii) translates into the generating function definition,

$$\sum_{l=0}^{\infty} A_l^{\alpha-1} \sigma^l = \frac{1}{(1-\sigma)^{\alpha}}, \quad \sigma = e^{-t}, \quad t > 0. \quad (44)$$

Property (iv) is the most interesting one as it concerns the case when α is a negative integer which is usually excluded (*e.g.* Chapman, [46]). Looking at (43), the quantity of interest is,

$$\lim_{\alpha \rightarrow -k=0, -1, \dots} \frac{\Gamma(l-\alpha)}{\Gamma(l+1)\Gamma(\alpha)} = (-1)^l \frac{\Gamma(k+1)}{\Gamma(l+1)\Gamma(k-l+1)} = (-1)^l \binom{k}{l}. \quad (45)$$

This also follows quickly from the binomial expansion of (44) and one could use generating functions systematically, *e.g.* Jordan, [47], to re-express the following remarks.⁹

I recall now the expression for the k th difference, *e.g.* Boole, [20], applied to the Kronecker delta, $\delta_l^{l'}$, considered as a function of l (*cf* Tauber and Dean, [50], Traub, [49]¹⁰). The expressions are,

$$\begin{aligned} \overrightarrow{\Delta}^k \delta_l^{l'} &= \sum_{m=0}^k (-1)^m \binom{k}{m} \delta_{k-m+l}^{l'} = (-1)^{l'-l-k} \binom{k}{l'-l} \\ \overleftarrow{\Delta}^k \delta_l^{l'} &= \sum_{m=0}^k (-1)^m \binom{k}{m} \delta_{-m+l}^{l'} = (-1)^{l-l'} \binom{k}{l-l'} \end{aligned} \quad (46)$$

⁹ The isomorphic algebraic scheme of generalised sequences, *e.g.* [48], [49], could also be employed.

This forms an operator calculus in the field of finite differences.

¹⁰ There appears to be an overall sign error in equations (3.6) and (3.7) of [50].

where $\overrightarrow{\Delta}$ is the forward difference and $\overleftarrow{\Delta}$ the backwards one. Comparing with (45) the conclusion is that,

$$\lim_{\alpha \rightarrow -k} A_{l-l'}^{\alpha-1} = \overleftarrow{\Delta}^k \delta_l^{l'} = (-1)^k \overrightarrow{\Delta}^k \delta_{l'}^l$$

which is the finite difference counterpart of property (iv), as I wished to show.

Unlike the continuous version, the higher difference $\overleftarrow{\Delta}^k \delta_l^0$ is not concentrated at the origin, $l = 0$, but extends to the right for a ‘distance’ k , emphasising the non-locality, or fuzziness, in its construction. As a simple feature, I remark that, $\sum_{l=0}^k \overleftarrow{\Delta}^k \delta_l^0 = \delta_k^0$, which just states that the strength of the (oscillating) ‘curve’ of a higher derivative is zero. The corresponding forwards quantity, $\overrightarrow{\Delta}^k \delta_l^0$, extends a distance k to the *left* of the origin.

As usual, employing forwards or backwards differences involves a manifest loss of symmetry which can be restored by shifting the origin of the graphs to their mid points. Analytically this is accomplished by the translation operator, conventionally written $\mathbf{E}^{\pm 1/2}$, and corresponds to using the central difference, δ , so that $\delta^k \delta_l^0$ is the closer analogue of the Dirac derivative $\delta^{(k)}(x)$.¹¹

Pictures of some discontinuous approximations to δ -function derivatives obtained by central differences of the (piecewise) continuous step function can be seen in van der Pol and Bremmer, [51], p.83.¹²

6. Discussion

I have given several disparate pieces of analysis based on the explicit calculation of a denominator, or restricted partition. On the way a neat finite difference expression, (12), for Euler’s functions was found and also one for the Todd operator, (14). An elegant derivation of a derivative expansion is likewise given.

An expression for the general wave W_q is given, (28), but this is not necessarily in its final form.

The expansion of the denominator into waves can be substituted into the Barnes ζ -function, (3), although would not aid its specific computation.

Brioschi’s early 1857 calculation of the simplest case of prime components is resurrected and leads to Dedekind sums, which is known more recently.

¹¹ There is, of course, an *intrinsic* non-locality in the continuous higher derivative.

¹² If the blocks there depicted are squashed to their midpoints, and $\epsilon = 1/2$, then these graphs yield precisely the symmetrical constructions, $\delta^k \delta_l^0$, here.

Appendix A. Proof of Brioschi's formula

For completeness, I give a pedagogic derivation of equation (29) for the wave in the simplest case when the components are all prime. There is nothing new in this, *cf* Brioschi in 1857, [34].

The starting point is always Euler's generating function for the denominator,

$$\sum_{l=0}^{\infty} \frac{l;}{\mathbf{d};} z^l = \frac{1}{\prod_i (1 - z^{d_i})} = \frac{1}{(1 - z^\alpha) \prod_i (1 - z^{\beta_j})} \equiv \frac{1}{(1 - z^\alpha) f(z)}. \quad (47)$$

where α is one component selected from the d_i and β_j the rest. Notationally I set $\alpha = q \in \mathbb{Z}$. Extracting the power z^l using residue calculus gives

$$\frac{l;}{\mathbf{d};} = \frac{1}{2\pi i} \oint_C dz \frac{1}{z^{l+1}} \frac{1}{(1 - z^q) f(z)} \quad (48)$$

where C circles the origin. Now blow up C to wrap round the other poles, which all lie on the unit circle and are given by the vanishing places of the denominator in (47). In this sum, I consider only those arising from the vanishing of the separated typical factor $(1 - z^q)$ at all the *non-trivial* q th roots of unity, $\rho_1, \dots, \rho_{q-1}$. This particular sum equals, by definition, the wave, W_q . The contour then continues on to infinity, which contributes nothing.

The residue of $1/(1 - z^q)$ at $z = \rho_j$ is easily found to be $-\rho_j/q$ so that the residue of the integrand is $\rho_j^{-l}/q f(\rho_j)$, and I thence obtain (29).

The multiple pole at $z = 1$, coming from all factors in the denominator would give the first (non-periodic) wave, W_1 , a polynomial, essentially a generalised Bernoulli polynomial, *e.g.* [9]. There are no other multiple poles, which accounts for the simplicity of this evaluation.¹³ The total denominator is then the sum

$$\frac{l;}{\mathbf{d};} = W_1 + \sum_{i=1}^d W_{d_i}. \quad (49)$$

If the components are only coprime, the roots ρ_i , for *each* component, separate into the prime roots for every factor of that component so giving a subset of waves, one wave, of the form (29), for each divisor. The total denominator is still given by (49) with the subsum,

$$W_{d_i} = \sum_{q/d_i} W_q. \quad (50)$$

¹³ Because of the separation of the roots, one might say that there is no interference between the waves.

The general case, when some components possess common factors, involves coincident poles on the unit circle originating from different factors on the denominator. These, like the poles at $z = 1$, give rise to polynomials. Analysing and organising this situation leads to Sylvester's theorem, [34].

Ehrhart, [52], Theorem 9.2, also gives the coprime expression for the denominator ('compteur') which he seems to derive independently. He uses it numerically to compute the denominator by calculating the expression to an adequate approximation. The bulk of the value comes from the first wave, polynomial part.

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